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Sparse hamiltonian 2-decompositions together with exact count of numerous Hamilton cycles

Zdzisław Skupień

Department of Discrete Math, Faculty of Applied Mathematics, AGH University of Science and Technology, al. Mickiewicza 30, 30–059 Kraków, Poland

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ABSTRACT

We construct multigraphs of any large order with as few as only four 2-decompositions into Hamilton cycles or only two 2-decompositions into Hamilton paths. Nevertheless, some of those multigraphs are proved to have exponentially many Hamilton cycles (Hamilton paths). Two families of large simple graphs are constructed. Members in one class have exactly 16 hamiltonian pairs and in another class exactly four traceable pairs. These graphs also have exponentially many Hamilton cycles and Hamilton paths, respectively. The exact numbers of (Hamilton) cycles and paths are expressed in terms of Lucas- or Fibonacci-like numbers which count 2-independent vertex (or edge) subsets on the n-path or n-cycle. A closed formula which counts Hamilton cycles in the square of the n-cycle is found for $n \geq 5$. The presented results complement, improve on, or extend the corresponding well-known Thomason's results.

1. Introduction

The word *decomposition* stands for an edge decomposition and *k-decomposition* comprises k parts. Decompositions are called *hamiltonian* and *traceable* if parts are exclusively Hamilton cycles and Hamilton paths, respectively. Hamiltonian (resp. traceable) 2-decompositions are named *hamiltonian* (resp. *traceable*) *pairs*. Symbols h_2 and t_2 stand for graphical functions, each value of which is a *number of hamiltonian pairs* and *number of traceable pairs*, respectively. In general, graph stands for a simple graph, and a multigraph is loopless.

The following result is of basic importance for what follows.

Theorem A (*Thomason [15]*). *Let G be a multigraph that has a hamiltonian pair and order three or more. Then the number, $h_2(G)$, of hamiltonian pairs in G is even and at least four. Moreover, for any two edges of G, the number of hamiltonian pairs in which the two edges are in the same part is also even.*

Inspiration came from C.A.B. Smith's classical theorem in [14]: *In any cubic graph the number of Hamilton cycles containing a given edge is even*. Moreover, Theorem A answers Sloane's question [11] if a hamiltonian 2-decomposable multigraph has another hamiltonian pair, see also Bosák [2, 11.44–11.55] for an extensive review of related problems and results.

The related problem concerning 2-decompositions into paths was originated by Ken Smith [13] within GRAPHNET on the Internet in 2001. Let p_2 denote the *number of decompositions of a graph into two nontrivial paths*. Introducing a parameter U and parity question for p_2 has resulted in the following specification of the two-path problem.

Theorem B (*[8]*). *Let G be a connected multigraph that is decomposable into two nontrivial paths whose set of endvertices is denoted by U. If $G \neq P$ (path) and $G \neq C$ (cycle), then G is the union of a different pair of edge-disjoint paths with the same set U of endvertices. In fact, the number $p_2(G, U)$ of such path decompositions is even unless G is homeomorphic with the graph of the symbol $+$ or $\perp$ (the latter being the claw $K_{1,3}$), because then $p_2 = 3$.*

E-mail address: skupien@agh.edu.pl.

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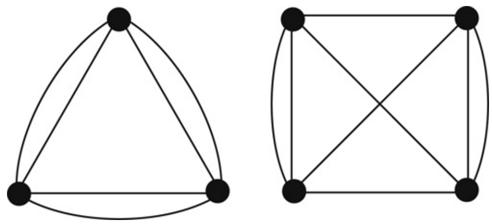

Fig. 1. Multigraphs M_3 and M_4.

Thus among multigraphs with maximum vertex degree 3 or 4, only all trees obtainable by subdividing stars $K_{1,3}$ or $K_{1,4}$, with set U comprising all (exactly) four odd-degree vertices, have odd number of decompositions, just 3, into two nontrivial paths. For related information, see [8]; see [4, Feb. 16–21, 2001] for the complete account of a brief but intense internet discussion on the two-path conjecture.

Our main results follow.

Theorem 1. *For each integer $n \geq 3$, there is an n-vertex (4-regular) multigraph M_n with exactly four hamiltonian pairs.*

Theorem 2. *For each $n \geq 3$, there is an n-vertex multigraph N_n which has exactly two traceable pairs.*

Theorem 3. *Each multigraph $M \neq C_2$ has an even number of traceable pairs.*

Theorem 4. *For each $n \geq 5$, there is a simple n-vertex graph with exactly four traceable pairs in which endvertices of the Hamilton paths make up a 3-element set.*

Analogous examples with 2-element sets of endvertices are presented by Thomason [15, graphs G_i, under Fig. 5] for each order $n \geq 6$ only.

Theorem 5. *For each integer $n \geq 9$ there is a simple graph on n vertices which has precisely 16 hamiltonian pairs.*

Thomason's related result [15] is that a 4-regular n-vertex graph with exactly 32 hamiltonian pairs exists for each $n \geq 10$.

Thus each of our Theorems, T, is closely related to Thomason's results and provides sharpness ($T = 1$), improvement ($T = 5$), or otherwise an extension of Thomason's. Theorem 3 follows from Theorem B but deserves an independent proof.

We also prove that some of multigraphs referred to in Theorems 1 and 2 make up two sequences, one with exponentially many Hamilton cycles and another with exponentially many Hamilton paths, respectively. Also sequences of simple graphs referred to in Theorems 4 and 5 are proved to have same properties. For counting Hamilton cycles in the squared cycle, see Section 5. The *number of Hamilton cycles* and that *of Hamilton paths* are counted by functions h and t, respectively. Additionally, given two vertices x, y, *Hamilton x–y paths* are counted by the function t_{xy}.

2. Constructions and proofs

Following Bosák's convention, the symbol ${}^2G$ stands for the multigraph which results on *doubling* each edge of a graph G. Similarly, ${}^2xy$ denotes the *double edge* joining vertices x, y.

In order to prove Theorems 1 and 2 we construct required multigraphs M_n and N_n, M_n being constructed recursively. (For a nonrecursive construction which singles M_n' (see Section 4 below) out of M_n's, see [8].) Firstly $M_3 := {}^2K_3$, the double triangle (with doubly adjacent pairs of vertices) in Fig. 1.

Inductive (or basic) assumption (to be checked at the general step of the construction) follows: For $n \geq 3$, we are able to introduce labels u_n, v_n, and w_n of three vertices of M_n such that u_n and v_n are doubly adjacent, i.e., ${}^2(u_nv_n) \subset E(M_n)$, and both are adjacent to w_n.

Let (on counting multiplicities) the remaining (fourth) neighbors of u_n and v_n be denoted respectively by t and z_n (whence $t = z_n = w_n$ if $n = 3$; $t = z_n \neq w_n$ if $n = 4$). Assume that

$$N_n = M_n - {}^2(u_nv_n).$$

Let M_{n+1} be obtained from M_n by putting $w_{n+1} := v_n$, $z_{n+1} := w_n$, and by splitting u_n into two degree-2 vertices, say u_{n+1} with neighbors t, w_{n+1} and v_{n+1} with neighbors z_{n+1}, w_{n+1} (thus N_{n+1} results, cf. Fig. 2), and next by joining the two new vertices by a pair of (parallel) edges, namely by ${}^2(u_{n+1}v_{n+1})$. This construction preserves the inductive assumption. Each M_n is 4-regular and, for $n \geq 4$, has exactly two pairs of parallel edges, the pairs are vertex-disjoint, and endvertices of either pair have a common neighbor which, for $n \geq 6$, is private, that is, different from that for the other pair. Moreover, M_n is unique up to isomorphism for $n \leq 6$ only.

It follows that each hamiltonian pair of M_n includes exactly one traceable pair of N_n and each traceable pair of N_n comprises edge-disjoint Hamilton u_n–v_n paths. Therefore each traceable pair of N_n extends to two hamiltonian pairs of M_n. This establishes the following observation.

$$N_n = M_n - {}^2(u_n v_n).$$

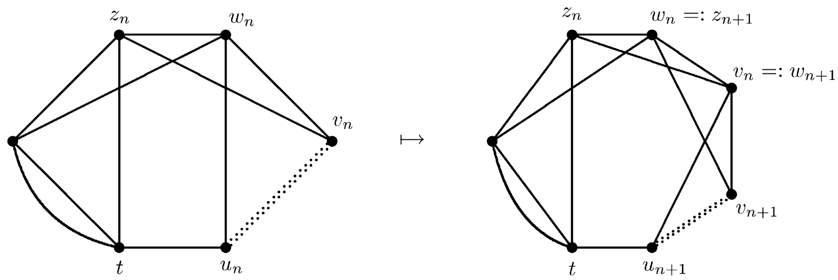

Fig. 2. $M_n \mapsto M_{n+1}$ for $n = 6$.

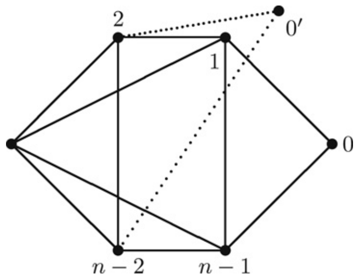

Fig. 3. The graph $G_6 := G_7' - \{0'\}$.

Proposition 6. *The number of hamiltonian pairs of M_n is twice the number of traceable pairs of N_n.* □

Proof of Theorem 2. Proceed by induction on n. N_3 clearly has two traceable pairs. Assume that so does N_n. Because each decomposition part of N_n is a Hamilton u_n–v_n path, the adjacent edges incident to either of endvertices v_n and u_n are in distinct parts. For the same reason the path $u_n w_n v_n$ cannot be in one part. Therefore edges $w_n v_n$ and tu_n are in one part, $w_n u_n$ and $z_n v_n$ in the other. By the construction, N_{n+1} arises on splitting the vertex u_n into v_{n+1} and u_{n+1}, see Fig. 2. Then the decomposition parts of N_{n+1} arise by uniquely extending those of N_n; namely, one new decomposition part includes the path $w_n v_n v_{n+1}$ and the edge tu_{n+1}, while the other one $w_n v_{n+1}$ and the path $z_n v_n u_{n+1}$. Thus the number, two, of traceable pairs remains unchanged on passing on from N_n to N_{n+1}. □

Corollary 7. *When restricted to multigraphs M_n and N_n constructed above, Theorems 1 and 2 are mutually equivalent.* □

Proof of Theorem 3. Assume that M has a traceable pair. If order of M is two then $M = C_2$ and $t_2 = 1$. Otherwise M has three or more vertices and the set, U, of endvertices of the two paths in the traceable pair has 2, 3 or 4 elements (which are the only vertices in M of degree less than 4). Let $\hat{M}_{|U|}$ be a 4-regular multigraph which includes M and either includes the set, E_2, of two new edges joining vertices in U if $|U| \le 3$ (new edges being parallel if $|U| = 2$) or includes four new edges which join a new vertex to each of four vertices in U. It is easily seen that $t_2(M) = h_2(\hat{M}_4)$ if $|U| = 4$. For $|U| = 3$, $t_2(M) = h_2(\hat{M}_3) - \hat{h}_2$ where $\hat{h}_2$ is the number of hamiltonian pairs in $\hat{M}_3$ with the set E_2 belonging to one cycle in each pair. For $|U| = 2$, $t_2(M) = \frac{1}{2} h_2(\hat{M}_2) = \hat{h}_2'$ where $\hat{h}_2'$ is the number of hamiltonian pairs in $\hat{M}_2$ such that one cycle in each of pairs includes two fixed adjacent edges, one from M and another one from E_2. Thus our theorem follows from parity results by Thomason. □

Let T_n, $n \ge 5$, be the 4-regular n-vertex graph comprising a cycle C_n together with all n chords of length two, the length being measured along the cycle. Thus $T_n = C_n^2$, the square of the cycle C_n. Assume that $0, 1, \ldots, n-1$ are consecutive vertices on C_n in the graph T_n. Let G_{n+1}' be obtained by splitting the vertex 0 into vertices 0 and $0'$ such that $0'$ is adjacent to 2 and $n-2$ only and 0 remains adjacent to 1 and $n-1$ only. Let G_n be obtained from T_n by removal of two adjacent chords,

$$G_n = T_n - \{0\text{–}2, 0\text{–}(n-2)\}, \quad n \ge 5,$$

whence $G_n = G_{n+1}' - \{0'\}$, cf. Fig. 3, graphs G_n' with $n \ge 6$ being constructed by Thomason. Thomason proves that T_n has exactly $2\lceil n/2 \rceil$ hamiltonian pairs, that is,

$$h_2(C_n^2) = 2\lceil n/2 \rceil = \mathrm{even}(n, n+1), \tag{1}$$

and notes that only four of pairs have any two *fixed* adjacent chords in distinct cycles. Hence, for $n \ge 9$, most pairs have those fixed adjacent chords in one of the Hamilton cycles. Consequently, $t_2(G_n') = 4 = t_2(G_n)$. This proves Theorem 4.

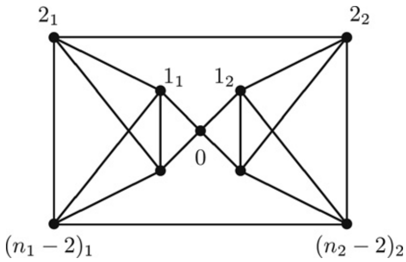

Fig. 4. The graph Γ_n with $n = 9$ and $n_1 = n_2 = 5$.

Table 1
(referred to in Section 3).

n	0	1	2	3	4	5	6	7	8	9
j_n^{**}	0	0	0	0	1	1	1	2	3	
j_n^{*}	0	1	1	1	2	3	4	6	9	
j_n	1	2	3	4	6	9	13	19	28	41

Graphs Γ_n. For $n \geq 9$, let G_{n_i} with $i = 1, 2$ be two graphs (constructed above) which are disjoint and satisfy the condition $n+1 = n_1 + n_2$. Use subscript i in order to differentiate between vertex names in the two G's. Let Γ_n be the 4-regular graph obtained from the union of the two G's by identifying vertices 0_1 and 0_2 into a new vertex 0 and by adding two new edges 2_1– 2_2 and $(n_1-2)_1$–$(n_2-2)_2$, cf. Fig. 4.

Proof of Theorem 5. Since the removal of both new edges from Γ_n results in a 1-connected graph, each cycle among hamiltonian pairs in Γ_n comprises exactly one new edge, say e, together with a pair of edge-disjoint paths such that both paths are in traceable pairs of distinct graphs G_{n_i} and each path shares an endvertex with the edge e. Hence $h_2(\Gamma_n) = t_2(G_{n_1}) \cdot t_2(G_{n_2}) = 4^2$. $\square$

Note that Thomason's simple graphs of order $n \geq 10$ with 32 hamiltonian pairs are obtained by bijectively identifying degree-2 vertices of G'_{n_1} with those of G'_{n_2} if $n = n_1 + n_2 - 2$ with both $n_i \geq 6$.

3. Auxiliary Fibonacci-like recurrence

The following Fibonacci-like recurrence relation plays a crucial role in what follows.

$$\varphi_n = \varphi_{n-1} + \varphi_{n-3}. \tag{2}$$

A vertex subset S of a graph is called to be 2-*independent* if any two vertices in S are at distance greater than two. Let j_n stand for the *number of* 2-*independent* vertex sets S on the path P_n with n vertices. Similarly, let j_n^* and j_n^{**} be the *numbers* of the sets S with additional property that each S includes respectively a fixed endvertex and both endvertices of P_n. In Table 1 all values of j_n^{**} and remaining ones for $n \leq 3$ are found by inspection. Clearly $j_n^* = j_{n-3}$ and $j_n^{**} = j_{n-3}^*$ for $n \geq 3$. Moreover, $j_n = j_n^* + j_{n-1}$ whence (2) holds for $\varphi_n = j_n$,

$$j_n = j_{n-1} + j_{n-3} \quad \text{for } n \geq 3. \tag{3}$$

Therefore $j_n^* = j_{n-3} = j_{n-3}^* + j_{n-4} = j_{n-3}^* + j_{n-1}^*$, i.e., (2) with $\varphi_n = j_n^*$ holds for $n \geq 4$ and also for $n = 3$ due to Table 1. Moreover, $j_n^{**} = j_{n-3}^* = j_{n-4}^* + j_{n-6}^* = j_{n-1}^{**} + j_{n-3}^{**}$ for $n \geq 6$. Hence and by Table 1, if $\varphi_n = j_n^{**}$ then (2) holds for $n \geq 5$.

Remark 3.1. The characteristic equation of the recurrence (2),

$$x^3 = x^2 + 1, \tag{4}$$

has one real root α (>1) and two mutually conjugate complex roots $\rho \exp(\pm i\theta)$ with $\rho = 0.826^+$. Hence, on assuming that $s = *, **$ or s is the empty symbol, the following general real solution of the recurrence (2) follows.

$$\varphi_n = j_n^s = A_1^s \alpha^n + A_2^s \rho^n \cos n\theta + A_3^s \rho^n \sin n\theta \quad (\sim\ A_1^s \alpha^n \quad \text{as} \quad n \to \infty).$$

The initial conditions in Table 1 for j_n^s determine the above constants A_k^s. Let $|s|$ be the number of stars $*$ in s, $|s| = 0, 1, 2$. Then

$$\alpha = 1.46557^+, \qquad A_1 = 1.3134^+, \tag{5}$$

$$j_n \sim A_1 \alpha^n, \quad \text{and} \quad j_n^s \sim (\alpha^{-3|s|} A_1)\alpha^n \quad \text{as } n \to \infty \tag{6}$$

since $j_n^* = j_{n-3}$ and $j_n^{**} = j_{n-6}$, whence $A_1^s = \alpha^{-3|s|} A_1$.

Table 2
(referred to in Theorem 8 and Proposition 9).

n	3	4	5	6	7	8	9	10
$h(M'_n)$	8	9	12	16	21	29	41	58
$t_{uv}(N'_n)$	4	4	5	7	9	12	17	24

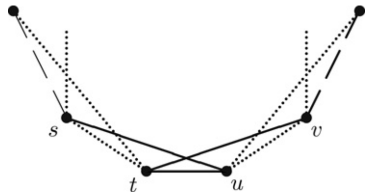

Fig. 5. Double reversal at the edge tu.

Remark 3.2. If x_1, x_2, x_3 are characteristic roots then any x_k^n and therefore the sum $\sigma_n := \sum_k x_k^n$ satisfy the recurrence (2), e.g.,

$$\sigma_n = \sigma_{n-1} + \sigma_{n-3} \quad \text{for } n \geq 3, \text{ with } \sigma_0 = 3, \sigma_1 = 1 \text{ and } \sigma_2 = \sigma_1^2 = 1 \tag{7}$$

by Viète's formulas, see Table 3 for more values.

4. Hamiltonian decompositions and the number of Hamilton cycles

The topic is originated by Shen Lin's question [5] (1965) if every 4-regular graph which is hamiltonian decomposable has another Hamilton cycle. An affirmative answer of Sloane [11] is generalized by Ninčák's result [6,12] that a multigraph with w edge-disjoint Hamilton cycles has at least $(2w-1)w$ Hamilton cycles, the bound being attained in the multigraph of order two and size $2w$. Thomason shows in [15, Corollary 2.2] that if a $2w$-regular multigraph G with $w \geq 2$ and of order at least three has a hamiltonian decomposition then G has at least $(3w-2)w$ Hamilton cycles and at least $(3w-2)(3w-5)\cdots 7\cdot 4$ hamiltonian decompositions, the number of such decompositions being even if G is 4-regular (or $w = 2$).

However, both bounds for $w \geq 3$ and one only which concerns Hamilton cycles for $w = 2$ are rather far below exact bounds. In particular, it seems that the bounds can be increasing together with order of the multigraph. A part of this statement, the one for $w = 2$ only, is supported by our exact count of Hamilton cycles among some n-vertex multigraphs M'_n chosen from among M_n's with exactly four hamiltonian pairs.

Let $M'_n = M_n$ for $n = 3, 4$ with u, v being any two doubly adjacent vertices. For $n \geq 5$, let s, t, u, v be four consecutive vertices on the n-cycle C_n. Then

$$M'_n := C_n^2 - \{su, tv\} + \{\text{single copies of } st, uv\}, \quad n \geq 5,$$
$$N'_n := M'_n - {}^2(uv) \quad \text{for } n \geq 3.$$

Note that M'_n coincides with M_n constructed in Section 2 above if the two double edges of M_n are adjacent to a common edge tu; it is so for M_n and M_{n+1} in Fig. 2.

Recall that $t_{uv}(N'_n)$ denotes the number of Hamilton u–v paths of N'_n.

Theorem 8. *For the number, $h(M'_n)$, of Hamilton cycles of M'_n, see Table 2 and*

$$\begin{aligned} h(M'_n) &= j_{n-1} + j_{n-4} + 4 \quad \text{for } n \geq 4, \\ &\sim A_1(\alpha^{-1} + \alpha^{-4})\,\alpha^n \approx 1.18 \times 1.46557^n \quad \text{as } n \to \infty. \end{aligned} \tag{8}$$

Proof. The result is easily seen for $n = 3, 4$. Let $n \geq 5$. In general, each multigraph M'_n has exactly 4 hamiltonian pairs which comprise 8 Hamilton cycles. Then four of the cycles avoid each length-2 chord of C_n and each of the remaining four cycles includes all length-2 chords of C_n, which appear in M'_n, but avoids the edge tu. In fact, any Hamilton cycle which includes both tu and a chord f is not in any hamiltonian pair because then $M'_n - \{tu, f\}$ has a cut vertex and is nonhamiltonian.

Let H be a Hamilton cycle which contains the edge tu. Cycles H (inclusive of the four from hamiltonian pairs) can be counted in terms of numbers j_k. Namely, suppose that a length-2 chord f belongs to the cycle H. Then H includes a path fef' which comprises an edge e of C_n and another chord f' such that endvertices of f and f' intertwine on C_n. Later on the path fef' will be called to be a double reversal at the edge e, cf. Fig. 5. In our case $e \neq tu$. Therefore all such edges e on $P_{n-2} \cap H$ where $P_{n-2} := C_n - \{t, u\}$ make up a 2-independent set in the line graph $L(P_k)$ of an appropriate subpath P_k of P_{n-2}, $k \leq n-2$. Just below we shall use the fact that $L(P_k)$ is a path, P_{k-1}, of order one smaller than that of P_k. Depending on the order n, there are up to three cases possible. Namely, for the two pairs of parallel edges in M'_n with $n \geq 5$,

(i) H avoids both pairs. This is possible only if $n \geq 7$. Then $k = n - 2$ and $(\#H)$ is j^{**}_{n-3} for $n \geq 5$ (which is j^{*}_{n-6} for $n \geq 6$, and is j_{n-9} for $n \geq 9$);
(ii) H avoids one pair (in two ways) and intersects another pair in two ways. Then $k = n - 3$ and $(\#H)$ is $4j^{*}_{n-4}$ for $n \geq 5$ (which is $4j_{n-7}$ for $n \geq 7$);
(iii) H intersects both pairs (in four ways). Therefore $k = n - 4$ and $(\#H) = 4j_{n-5}$ for $n \geq 5$, the four cycles H from the hamiltonian pairs being included in the count.

Hence on using Table 1 we get $h(M'_n)$ as in Table 2 and in formula (8) for $n = 5, 6, 7, 8$. Moreover, for $n \geq 9$,

$$h(M'_n) = 4 + j_{n-9} + 4j_{n-7} + 4j_{n-5}$$

whence, on using recurrence (3) five times, we get (8), asymptotics being derived from (6) and (5). □

Proposition 9. *The number of Hamilton u–v paths joining degree-2 vertices in N'_n is*

$$t_{uv}(N'_n) = j_{n-4} + j_{n-5} + 2 \quad \text{for } n \geq 5,$$
$$\sim A_1(\alpha^{-4} + \alpha^{-5})\,\alpha^n \approx 0.479 \times 1.46557^n \quad \text{as} \quad n \to \infty.$$

with initial values given in Table 2, see also Table 1.

Proof. Values of $t_{uv}(N'_n)$ in Table 2 for $n = 3, 4$ are found by inspection. From the proof of the preceding theorem, for $n \geq 5$, we get $t_{uv}(N'_n) = 2 + j^{*}_{n-4} + 2j_{n-5}$ whence the stated results follow. □

The multigraph N'_n has many more Hamilton paths. For instance, the number of Hamilton paths of N'_n all of which are sections of Hamilton cycles of the supermultigraph M'_n equals $2j^{*}_{n-2} + nj^{*}_{n-3} + nj^{*}_{n-4} + 2$ which, for $n \geq 4$, is greater than and is growing much quicker than $t_{uv}(N'_n)$. Additionally, for all but a finite number of pairs of nonadjacent vertices x, y, there are Hamilton x–y paths in N'_n. The number of those additional paths grows exponentially, too.

5. The case of simple graphs

We first count Hamilton cycles in the square, C^2_n, of the cycle C_n in case $n \geq 5$. Then the square is a 4-regular graph. Let tu be an edge of C_n. The path including tu and an adjacent chord of C_n in C^2_n, joining either endvertex (t or u) of tu to the neighbor on C_n of the other endvertex of tu, is said to be a *reversal* at tu. Let s, t, u, v be four consecutive vertices of C_n in C^2_n. Then the path $sutv$ of C^2_n is called a *double reversal* at the edge tu, see Fig. 5. The path comprising three dotted edges incident to vertices s, t in Fig. 5 is said to include a *single reversal* at the edge st.

Given an edge e of C_n, there is a unique Hamilton cycle of C^2_n, denoted by $H(e)$ or simply H, which avoids the double reversal at e, that is, H avoids all three edges of the double reversal. In fact, if the cycle H avoids the double reversal $sutv$, i.e., $H = H(tu)$, then H comprises both the edges neighboring tu on C_n, namely st and uv, together with all chords which do not belong to the double reversal. Hence H does not include any double reversal. Moreover, any such cycle H $(= H(e))$ is uniquely characterized by requiring that H be Hamilton cycle including a specified single reversal. For instance, if $e = tu$ in Fig. 5, the specified single reversal is to comprise dotted edges incident to t (or to u). The uniqueness of H in this sense is crucial for counting all Hamilton cycles (or Hamilton paths) in what follows. Let $H'(e)$ denote the complement of $H(e)$ in C^2_n. Clearly, $H'(e)$ is a Hamilton cycle and $H'(e)$ includes exactly one double reversal. It is Thomason who notes that, for even n, all hamiltonian pairs of C^2_n are of the form $H(e)$ and $H'(e)$ with e ranging over all n edges of C_n. For odd n, one more hamiltonian pair is made of C_n and the cycle comprising all chords of C_n. (This proves formula (1), see above.) Therefore we conclude that each of remaining Hamilton cycles (if any exists) includes double reversals and no single reversal at edges of C_n. Let H' be a Hamilton cycle which includes a double reversal and let

$$E^*(H') = \{e \in E(C_n) \colon H' \text{ includes double reversal at } e\}.$$

It is easily seen that $E^*(H')$ is a 2-independent subset of $E(C_n)$ in the sense that any two elements $f, f' \in E^*(H')$ are separated by two or more other edges on either subpath of C_n joining the edges f, f'. Therefore cycles H' (inclusive of all $H'(e)$) can be counted in terms of numbers like j_k, cf. proof of Theorem 8.

Let γ_n denote the *number of* 2-*independent* (vertex or edge) *sets on the cycle* C_n, the empty set being included. By inspection,

$$\gamma_n = n + 1 \quad \text{for } n = 3, 4, 5. \tag{9}$$

Lemma 10. *The number $\gamma_n = \sigma_n = \sum_k x^n_k \sim \alpha^n \approx 1.46557^n$ as $n \to \infty$.*

Proof. From author's general result [10] it follows that γ_n satisfies recurrence (2),

$$\gamma_n = \gamma_{n-1} + \gamma_{n-3} \quad \text{for } n \geq 6,$$

with initial conditions (9). Hence, due to Remark 3.2, (7) and (9), σ_n is the extension of γ_n to lower arguments $0 \leq n < 3$ (Table 3). □

Table 3
(referred to in Remark 3.2 and in proof of Lemma 10).

n	0	1	2	3	4	5	6	7	8	9	10
σ_n ($\supset \gamma_n$)	3	1	1	4	5	6	10	15	21	31	46

Table 4
(referred to in Theorem 11).

n	5	6	7	8	9	10
$h(C_n^2)$	12	16	23	29	41	56

Theorem 11. *The number of Hamilton cycles of the square of the n-cycle is*

$$h(C_n^2) = \sigma_n + 2\lceil n/2 \rceil \quad \textit{for } n \geq 5, \tag{10}$$

see preceding Lemma for asymptotics and Table 4 for a few values.

Proof. The summand σ_n counts Hamilton cycles each of which either includes a double reversal or is C_n. The second term is the number of hamiltonian pairs, see formula (1), and is the number of the cycles $H(e)$ as above together with one more cycle formed of all chords of C_n in C_n^2 if n is odd. □

Using Cardano's formulas for roots x_k of Eq. (4) and abbreviations

$$D = \left(\frac{1}{2}(29 - 3\sqrt{93})\right)^{1/3} \quad \text{and} \quad E = \left(\frac{1}{2}(29 + 3\sqrt{93})\right)^{1/3},$$

we get roots

$$\alpha = x_1 = \frac{1}{3}(1 + D + E) \quad \text{and} \quad x_{2,3} = \frac{1}{6}\left(-(D + E - 2) \pm i(E - D)\sqrt{3}\right),$$

which readily gives closed formulas for $\sigma_n = \sum_k x_k^n$, γ_n and $h_2(C_n^2)$, e.g.,

$$\sigma_n = \alpha^n + 2\rho^n \cos(n\theta) \tag{11}$$

where $\rho := |x_{2,3}| = \frac{1}{3}\sqrt{D^2 + E^2 - D - E}$ and $\theta := \pi - \arctan \frac{(E-D)\sqrt{3}}{D+E-2}$.

We are going to show that there are exponentially many Hamilton paths in G_n and Hamilton cycles in Γ_n, the two graphs defined in Section 2 above. In order to fix notation, let s, t, u, v, w be consecutive vertices of the cycle C_n. Assume that $G_n = C_n^2 - \{su, uw\}$.

Lemma 12. *The number of Hamilton s–w paths joining degree-3 vertices in G_n is equal to* $t_{sw}(G_n) = 2\lfloor (n-3)/2 \rfloor$.

Proof. All those paths are obtainable by modifying a Hamilton cycle, H, from any of $2\lceil n/2 \rceil$ hamiltonian pairs of C_n^2 such that either H comprises all chords of C_n if n is odd or H avoids a double reversal at any edge on the s–w section of C_n, the section being $C_n - \{t, u, v\}$. Modification of H consists of taking $H - u$ and replacing therein the chord tv by the path tuv. Hence the number of cycles H equals $2\lceil n/2 \rceil - 4$, whence the result follows. □

Lemma 13. *The number of Hamilton u–w paths joining vertices of degree 2 and 3 in G_n equals* $t_{uw}(G_n) = j_{n-5} + j^*_{n-3} + 2$.

Proof. By the definition of G_n, each path to be counted is obtained by removing the chord uw from a Hamilton cycle of C_n^2 such that the cycle includes uw but avoids the chord us. The first two summands, j_{n-5} and j^*_{n-3} (with $j^*_{n-3} = j_{n-6}$ for $n \geq 6$), count the number of Hamilton cycles each of which includes the double reversal respectively at uv and at vw but together with edge ut. The last term, 2, is the number of Hamilton cycles which avoid the double reversal at st or at tu. □

As in the proof of Theorem 5, let Γ_n with $n \geq 9$ be obtained from two disjoint graphs G_{n_i} ($i = 1, 2$) with $n_1 + n_2 + 1 = n$, $n_i \geq 5$ and with vertices x_i where $x = s, t, u, v, w$ (as above) by identifying u_i and u_2 into a vertex u of Γ_n and by adding two new edges s_1s_2 and w_1w_2. Use Θ to denote the exact order of growth.

Theorem 14. *There are exponentially many Hamilton cycles in Γ_n because*

$$\begin{aligned} h(\Gamma_n) &= 2t_{u_1w_1}(G_{n_1}) \cdot t_{u_2w_2}(G_{n_2}) + 2t_{s_1w_1}(G_{n_1}) \cdot t_{s_2w_2}(G_{n_2}) \\ &= \Theta(\alpha^n) \quad \textit{as} \quad n \to \infty. \end{aligned}$$

Proof. Any Hamilton cycle $\tilde{H}$ of Γ_n includes either exactly one of new edges (in two symmetric ways) or both new edges but then $\tilde{H}$ passes through the new vertex u along t_iuv_i (in two ways, $i = 1, 2$). The first summand is exponential and the order of its growth follows from Lemma 13 and formulas (6) and (5). □

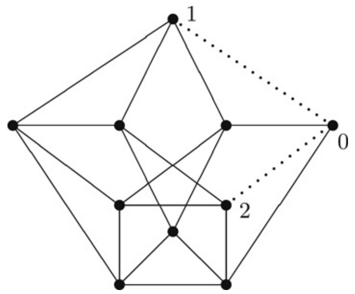

Fig. 6. Graph T_{10}^{*} and its subgraph G_{10}^{*}.

6. Concluding remarks

This paper is a sequel of the announcement [9]. What is new here is detailed counting of Hamilton paths or cycles. The most outstanding of new results is the closed formula (10) and (11) for the number of Hamilton cycles in the square of a cycle.

Conjecture. 16 *is the smallest value among* nonzero *numbers of hamiltonian pairs in* 4*-regular simple graphs of any order* ≥ 15.

Computer was used to check that among n-vertex connected 4-regular graphs the number of hamiltonian pairs, if nonzero, is at least $h_2(T_n) = 2\lceil n/2\rceil$ if $n \leq 10$. For $n = 9, 10$, all $16 + 59$ such graphs presented in [1] were checked. (Erratum: in item 32 for $n = 10$ on p. 34 of [1], replace 6–7,8 by 6–7,9.). Computations being organized as follows. Given a graph G, an edge e of G is chosen and only all Hamilton cycles, if any, which include the edge e are generated and counted. For each cycle, however, the cycle is removed if its complement in G is not a single cycle. Thus $h_2(G)$ is found. Next both extrema (or only minimum) of the function $G \mapsto h_2(G)$ over a fixed order n are (is) found (inclusive of all corresponding arguments G).

The author thanks Dr. Andrzej Żak and MSc Artur Fortuna [18] for checking the sharpness of the bound $h_2 \geq h_2(T_n)$ in the case $10 < n \leq 16$.

It appears that among graphs of order up to 10 item 55 for $n = 10$ in [1] is the only additional graph, denoted by T_{10}^{*} in Fig. 6, with h_2 equal to the lower bound. It can be shown that the graph $G_{10}^{*} := T_{10}^{*} - \{0\text{-}1, 0\text{-}2\}$ has $t_2 = 4 = t_2(G_{10})$. Thus G_{10}^{*} is a graph which can replace G_{10} in constructions of new graphs Γ_n of order $n \geq 14$ (with $h_2 = 16$).

Let us call graphs like G_n or G_{10}^{*} to be V-triples. A *V-triple*, denoted formally by $(G, \{b, c\}, a)$, is a graph G with all but three vertices a, b, c of degree 4, with a of degree 2, and both b and c of degree 3. The above computations are being extended in order to get V-triples of order n with (largest or) smallest nonzero t_2. Namely, for each G of order n and with $h_2 > 0$ only, and for each vertex x (ranging possibly over a set of single representatives of all vertex-similarity classes) in G, one of edges incident to x is denoted by e_x and remaining ones by $e_j, j = 1, 2, 3$. Then hamiltonian pairs determine a *characteristic* of x, say $\chi(x)$, which is a partition of $h_2(G)$, $h_2(G) = \sum_j h(x, e_j)$, where $h(x, e_j)$ is the number of hamiltonian pairs in which the edge pair $\{e_x, e_j\}$ (as well as the pair of remaining edges incident to x) is included in one Hamilton cycle. Then $G_{x,j}^{*} := G - \{e_x, e_j\}$ is a V-triple with $t_2 = h_2(G) - h(x, e_j), j = 1, 2, 3$. (For instance, $2 + 2 + 6$ is the characteristic of $x = 0$ in the graph T_{10}^{*} in Fig. 6, where summand 6 corresponds to dotted edges.) Hence in order to find V-triples with smallest t_2 one performs minimization over j, over $x \in V(G)$, and finally over G. Other V-triples can be obtained from multigraphs with up to two (adjacent) double edges.

Besides constructing multigraphs and graphs with smallest known numbers of hamiltonian or traceable 2-decompositions (which has been the main objective), another goal here is to suggest that requiring the existence of such a decomposition forces the existence of many Hamilton cycles and Hamilton paths, respectively.

One of Thomason's results [15, Corollary 1.2] implies the following generalization of the well-known theorem on cubic graphs (due to C.A.B. Smith, see Introduction). A multigraph with vertices of only odd degree has an even number of Hamilton cycles containing a given edge. Therefore a regular multigraph of odd degree either is nonhamiltonian or has 3 or more Hamilton cycles. In contrast, infinitely many 4-regular (or $2k$-regular with any $k \geq 2$) multigraphs with exactly one Hamilton cycle exist; examples are constructed by Fleischner [3], and the smallest example (with any $k \geq 2$) has 28 vertices.

Therefore it is worth recalling the long-standing, still open, Sheehan conjecture [7] that each simple 4-regular hamiltonian graph has more than one Hamilton cycle. If the conjecture is true then every regular simple graph, G, of even degree, say $2k$, $k \geq 2$, which has a Hamilton cycle, say H (and $G \neq H$), has k or more Hamilton cycles. Namely, it is Petersen's result that a regular graph G of even degree admits a decomposition into 2-factors. One of the 2-factors can clearly be the Hamilton cycle H. Therefore G would have at least k Hamilton cycles, one new "private" cycle would be ensured in each union of H and another (edge-disjoint) 2-factor from the decomposition. How specialized Sheehan's conjecture is is shown by C. Thomassen. He proves [16] that a hamiltonian m-regular graph has a second Hamilton cycle if $m \geq 300$. Moreover, he reports that his (probabilistic) method ensures the result for $m \geq 73$. However, he notes, too, that Sheehan's conjecture (i.e., the case of $m = 4$) cannot be proved by using that specific method.

Added in proof

Multigraphs M_n' (Section 4) were constructed by West [17], too.

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